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Topological Reduction of 4D SYM to 2D σ–Models

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By considering a (partial) topological twisting of supersymmetric Yang-Mills compactified on a 2d space with 't Hooft magnetic flux turned on we obtain a supersymmetric σ-model in 2 dimensions. For $N = 2$ SYM this maps Donaldson observables on products of two Riemann surfaces to quantum cohomology ring of moduli space of flat connections on a Riemann surface. For $N = 4$ SYM it maps S-duality to T-duality for σ-models on moduli space of solutions to Hitchin equations.

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1. Introduction

One of the main sources of insights into the dynamics of 4 dimensional quantum field theories comes from analogies with simpler 2 dimensional quantum field theories. It is the aim of this paper to make this analogy more precise in the context of supersymmetric gauge theories in 4 dimensions and special classes of supersymmetric σ-models in 2 dimensions. This will in particular allow us to map Donaldson observables on products of two Riemann surfaces to quantum cohomology ring of the moduli space of flat connections on one of the surfaces. In the context of $N = 4$ YM, this reduction allows us to map S-duality to T-duality of certain σ-models, thus relating electric-magnetic duality to momentum-winding duality of σ-models.

The basic idea is rather simple. We consider a Euclidean quantum field theory on a product of two Riemann surfaces $\Sigma \times C$ in the limit where the size of one of them, say C shrinks to zero. This gives rise to a quantum field theory on Σ. The reduction of 4d Yang-Mill theory to 2d is in general very complicated due to the fact that different regimes of field configurations of the 4d theory result in different 2d effective theories which are related to each other in a complicated way. For four dimensional gauge theories a single regime of field configuration can be singled out by restricting attention to the sectors of path integral with non-trivial 't Hooft magnetic flux on C (which thus avoid having reducible gauge connections).

Starting from supersymmetric quantum field theories in 4d, we expect to get a supersymmetric theory in 2d. This is only the case when C is a torus with periodic boundary conditions. In the case C is a torus, turning on the flux unfortunately leads to a trivial quantum field theory on Σ with all the degrees of freedom frozen out. However one can consider a topologically twisted version of the 4d theory, which does give rise to a non-trivial supersymmetric 2d theory for any choice of C with genus greater than 1. In fact we can consider a fully twisted topological theory in 4 dimensions giving rise to a topological σ-model in 2 dimensions or we can consider partial twisting of the 4 dimensional theory only along the C directions and obtain an untwisted supersymmetric σ-model on Σ. Each twisting has its virtue: The fully twisted version is useful in that the topological amplitudes in 4d, being independent of the size of C, are directly related to topological amplitudes of the σ-model in 2d. The partially twisted theory, on the other hand, even

though it depends on the size of C, carries more information about non-topological aspects of the 4d theory [1]. We will consider both twistings in this paper.

Let us first consider $N = 1$ supersymmetric theory. The manifold M^4 has a product structure and therefore the holonomy group is reduced to $U(1)_\Sigma \times U(1)_C$, where each $U(1)$ is the holonomy of the corresponding surface. The $U(1)$ charges of the supersymmetry generator is given by $(\pm\frac{1}{2}, \pm\frac{1}{2})$. In addition the supercharge carries an R charge ± 1 (which with an appropriate choice of $N = 1$ theories with matter is anomaly free) which is correlated with the chirality of the spinor (even or odd number of minus signs in their $U(1)_\Sigma \times U(1)_C$ charge). If we twist the $U(1)_C$ by adding $-R/2$ to it, we find that there are two components of the supersymmetry which become spin 0 in the C direction and are both of the form $(+\frac{1}{2}, 0)$. We thus end up with a $(2, 0)$ supersymmetric theory on Σ for arbitrary choice of C with genus greater than 1. If we had in addition twisted the $U(1)_\Sigma$ by adding $-R/2$ we would have obtained a topologically twisted $(2, 0)$ theory in 2d. With standard twistings, in the case of $N = 2$ theories the same construction leads to a $(2, 2)$, and for $N = 4$ it leads to a $(4, 4)$ supersymmetric theory on Σ. In this paper we will mostly concentrate on the case of pure $N = 2$ and $N = 4$ YM theory. Extension of these to $N = 1$ and to $N = 2$ theories with matter are presently under consideration.

2. Reduction

We now consider this reduction in more detail. Let us first concentrate on pure YM theory on a four-dimensional manifold M^4, which has a product structure $M^4 = \Sigma \times C$. Let us choose the metric on this manifold to be block diagonal $g = g_\Sigma \oplus g_C$, where g_Σ (g_C) is the metric on Σ (C). The YM connection can be decomposed into two pieces $A = A_\Sigma + A_C$, where A_Σ (A_C) is the component of A along Σ (C). To discuss the reduction to 2d let us rescale the metric $g_C \to \epsilon g_C$ on C. Under this transformation different terms in the action scale differently

$$S = \frac{1}{4e^2} \int_{M^4} \mathrm{Tr} \Big[\frac{1}{\epsilon} F_C \wedge *F_C + 2(d_C A_\Sigma - D_\Sigma A_C) \wedge *(d_C A_\Sigma - D_\Sigma A_C) + \epsilon F_\Sigma \wedge *F_\Sigma \Big] . \tag{2.1}$$

[1] A possible relation between the dynamics of the 4d supersymmetric theories and those of corresponding σ-models has been conjectured in ref.[1].

Operation $*$ is defined with respect to unrescaled metric $g_\Sigma \oplus g_C$ and $D_\Sigma = d_\Sigma - i[A_\Sigma, \cdot]$.

In the limit $\epsilon \to 0$ the first term in the action enforces the component A_C to be flat ($F_C = 0$), while the second term gives rise to the σ-model action. The last term produces the corrections of order $O(\epsilon)$ that are irrelevant in the limit $\epsilon \to 0$.

We will denote the moduli space of flat connections on C by $\mathcal{M}(C)$. In order to specify the flat connection A_C on $\Sigma \times C$ one should specify a map $X : \Sigma \to \mathcal{M}(C)$. In this notation the flat connection becomes

$$A_C(w, \bar{w}, z, \bar{z}) = A_C(w, \bar{w} | X(z, \bar{z})), \tag{2.2}$$

where $z, \bar{z}$ $(w, \bar{w})$ are complex coordinates on Σ (C).

The flatness condition $F_C = 0$ implies that operator D_C is nilpotent, $D_C^2 = 0$. The tangent space to the moduli space of flat connection $\mathcal{M}(C)$ is given by D_C cohomology $H^1(C, \mathcal{G})$. We will always choose representatives that satisfy harmonicity condition $D_C^\mu \alpha_\mu = 0$, which is just the gauge fixing condition. The variation of the flat connection δA_C can be decomposed with respect to some basis $\{\alpha^I\} \subset H^1(C, \mathcal{G})$ modulo the gauge transformation

$$\frac{\partial A_C}{\partial X^I} = \alpha_I + D_C E_I \tag{2.3}$$

where E defines the connection on the moduli space $\mathcal{M}(C)$ (similar construction appears in [2], see also Appendix A).

The moduli space of flat connections $\mathcal{M}(C)$ is a Kähler manifold with the Kähler form and the metric given as follows

$$\omega_{IJ} = \mathrm{Tr} \int_C \alpha_I \wedge \alpha_J \quad \text{and} \quad G_{IJ} = \mathrm{Tr} \int_C \alpha_I \wedge *\alpha_J \ . \tag{2.4}$$

It is convenient to use the complex coordinates X^i and $X^{\bar{k}}$ on $\mathcal{M}$.

The action (2.1) is essentially quadratic in A_Σ, ignoring the terms of order $O(\epsilon)$. Moreover the action does not depend on the derivatives of A_Σ with respect to the coordinates on Σ. Hence A_Σ plays the role of an auxiliary field. Therefore one can attempt to integrate out A_Σ. This can be done if the connection on C is irreducible, which would allow us to invert the Laplacian $D_{\overline{w}} D_w$ on C:

$$A_\Sigma = E_i \partial_\Sigma X^i + E_{\bar{k}} \partial_\Sigma X^{\bar{k}} \ . \tag{2.5}$$

If the gauge field on C is reducible the Laplacian has zero modes which would give rise to additional degrees of freedom on Σ (and in particular dropping the $O(\epsilon)$ terms in (2.1)

cannot be justified in such cases). These additional degrees of freedom are described by residual gauge theory on Σ. Moreover if the dimension of the residual gauge symmetry jumps as we move on $\mathcal{M}$ the resulting 2d theory on Σ would be very complicated. This happens for example if we consider flat $SU(N)$ gauge fields on C. However if we consider $SU(N)/Z_N$ gauge theory and restrict the path-integral to the subsector where we turn on a non-trivial 't Hooft magnetic flux on C, then the connection on C is irreducible for all $\mathcal{M}$, the gauge group is completely broken and A_Σ can be integrated out. We will mainly concentrate on this case, but comment about some aspects of the more general case below.

Substituting the flat connection A_C and the expression for A_Σ (eq. (2.5)) into the action (2.1) one gets the σ-model action of the standard form

$$S = \frac{1}{2e^2}\int_\Sigma d^2z\ G_{i\bar{k}}(\partial_z X^i \bar{\partial}_{\bar{z}} X^{\bar{k}} + \bar{\partial}_{\bar{z}} X^i \partial_z X^{\bar{k}})\ . \tag{2.6}$$

It is also easy to see that turning on the θ angle for the YM is equivalent to turning on a $B-$field in the direction of the Kähler class. In this way we see that $\tau = i/4\pi e^2 + \theta/2\pi$ is now playing the role of the complexified Kähler modulus of this σ-model[2].

The moduli space of holomorphic instantons for this σ-model can be shown to coincide [3] with the the moduli space of self-dual connections of the 4d YM theory in the limit $\epsilon \to 0$. In particular one can view anti-self-dual connections as holomorphic connections (whose curvature vanish in the $(2,0)$ and $(0,2)$ directions) which satisfy $g^{i\bar{j}}F_{i\bar{j}} = 0$. The latter condition in the limit $\epsilon \to 0$ becomes $F_C = 0$, whereas the holomorphicity of the connection is equivalent to holomorphic instantons of the 2d theory.

Now consider the dimensional reduction of the topological YM theory which is the twisted version of the $N = 2$ $d = 4$ supersymmetric Yang-Mills theory [4]. In this case one ends up with the $(2,2)$ supersymmetric σ-model on $\mathcal{M}$. It is convenient to formulate this model in the complex notation. In the bosonic sector of $N = 2$ SYM theory there is a scalar field ϕ in addition to Yang-Mills connection A. The fermionic fields are the following: a scalar η, a self-dual two form that can be decomposed to a scalar λ and $(2,0)$ and $(0,2)$ forms λ_{zw} and $\lambda_{\bar{z}\bar{w}}$, and a 1-form with the components χ_z, χ_w, $\chi_{\bar{z}}$, $\chi_{\bar{w}}$. Since

[2] To fix the gauge at the quantum level we have to introduce the gauge ghosts. In the semiclassical approximation in the limit $\epsilon \to 0$ the integration over quadratic fluctuations of the gauge field (orthogonal to zero modes) near the flat connection $A = A_C$ produces $\det^{-2}\Delta$, where Δ is the covariant Laplacian on C, while the integration over the ghost fields gives rise to the factor $\det\Delta$. Combining together the determinants modifies the action of two-dimensional σ-model.

the action is linear in fermionic fields λ, η, χ_z and $\chi_{\bar{z}}$, one can integrate them out. Such an integration gives rise to the following constraints: $D_w \chi_{\bar{w}} = 0$, $D_{\bar{w}} \chi_w = 0$, $D_w \lambda_{\bar{w}\bar{z}} = 0$, $D_{\bar{w}} \lambda_{wz} = 0$. These fields are cotangent to the moduli space $\mathcal{M}(C)$ of flat connections on C. Therefore in the basis $\alpha_{i\bar{w}}$, $\alpha_{\bar{k}w}$ they can be represented as linear combinations

$$\chi_{\bar{w}} = \chi^i \alpha_{\bar{w}i},\ \chi_w = \chi^{\bar{k}} \alpha_{w\bar{k}},\ \lambda_{\bar{w}\bar{z}} = \rho^i_{\bar{z}} \alpha_{\bar{w}i},\ \lambda_{wz} = \rho^{\bar{k}}_z \alpha_{w\bar{k}}, \tag{2.7}$$

where χ^i, $\chi^{\bar{k}}$, $\rho^i_{\bar{z}}$ and $\rho^{\bar{k}}_z$ are two dimensional fermionic fields on Σ. The action is also quadratic in scalar fields ϕ and $\bar{\phi}$ and does not depend on the derivatives of these fields with respect to coordinates on Σ (in the leading order $\epsilon \to 0$). Notice that self-interaction ϕ^4 is suppressed in the limit $\epsilon \to 0$. Therefore one can just solve the equations of motion for ϕ and $\bar{\phi}$

$$\phi = \chi^{\bar{k}} \chi^i \Phi_{\bar{k}i}, \quad \bar{\phi} = g^{z\bar{z}} \rho^{\bar{k}}_z \rho^i_{\bar{z}} \Phi_{\bar{k}i}, \tag{2.8}$$

where $\Phi_{\bar{k}i}$ is the curvature on the principal bundle on $\mathcal{M}$ (see Appendix A).

Similar to the above non-supersymmetric model we integrate over components A_Σ of the gauge connection. Semiclassically A_Σ is given by an expression (2.5) plus a bilinear combination of the fermionic fields. Thus the integration over the field ϕ and A_Σ results in a four-fermionic interaction in the Lagrangian.

At this stage, due to the underlying $(2,2)$ supersymmetry and having identified the target space one can directly write the σ-model action for the reduced theory. However it is instructive to check that it really results from this reduction. In particular in the quantum theory one has to introduce the gauge ghosts and integrate over the quadratic fluctuations near the solutions of the classical equations of motion. It is easy to check that the determinants of the Laplace operators cancel due to supersymmetry in contrast to the above non-supersymmetric case. Therefore, as expected, we get a supersymmetric twisted σ-model on $\mathcal{M}$ (**A** model) with the standard action[3] [5]

$$\begin{aligned} S = \frac{1}{e^2} \int_\Sigma \mathrm{d}^2 z\ \Big[G_{i\bar{k}} \big(\frac{1}{2} \partial_z X^i \bar{\partial}_{\bar{z}} X^{\bar{k}} + \frac{1}{2} \bar{\partial}_{\bar{z}} X^i \partial_z X^{\bar{k}} + i \rho^{\bar{k}}_z \overline{D}_{\bar{z}} \chi^i + i \rho^i_{\bar{z}} D_z \chi^{\bar{k}} \big) \\ - R_{i\bar{k}j\bar{l}} \rho^i_{\bar{z}} \rho^{\bar{k}}_z \chi^j \chi^{\bar{l}} \Big], \end{aligned} \tag{2.9}$$

where $D_z \chi^i = \partial_z \chi^i + \chi^j \Gamma^i_{jk} \partial_z X^k$, $\overline{D}_{\bar{z}} \chi^{\bar{i}} = \bar{\partial}_{\bar{z}} \chi^{\bar{i}} + \chi^{\bar{j}} \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\partial}_{\bar{z}} X^{\bar{k}}$.

[3] The **A** twisting is inherited from four dimensions. If we consider the partial twisting of the four dimensional theory described above, we would obtain the untwisted σ-model on $\mathcal{M}$.

The anomaly in the fermion number is the same for the original 4d topological theory and for the σ-model. In the case of $SU(N)$, in particular the $c_1(\mathcal{M}) = Nh_2$ (where $h_2 \in H^2(\mathcal{M}, \mathbf{Z})$), in accord with the $U(1)$ 'ghost' number violation for the $N=2$ $SU(N)$ theory.

The physical operators of this σ-model are given by the BRST cohomology. Let us consider the relation between the physical operators in σ-model and those of the 4d topological YM theory. It is easy to check that under dimensional reduction the local physical operator of ghost number 4 of the 4d topological theory $\mathcal{O} = \mathrm{Tr}\phi^2$ becomes

$$\mathcal{O} \to b = \chi^i \chi^{\bar{k}} \chi^j \chi^{\bar{l}} R_{i\bar{k}j\bar{l}} \in H^4(\mathcal{M}). \tag{2.10}$$

Actually $\mathrm{Tr}\phi^2 \to b$ is true only classically. As we will see later, at the quantum level they differ by a c-number. Its descendant, a non-local physical operator $\int_C \mathcal{O}^{(2)} = \mathrm{Tr} \int_C d^2w \; (F_{w\bar{w}}\phi + \chi_w \chi_{\bar{w}})$ with the ghost number 2 becomes a local operator which represents a Kähler class

$$I(C) = \int_C \mathcal{O}^{(2)} \to a = \omega_{i\bar{k}} \chi^i \chi^{\bar{k}} \in H^2. \tag{2.11}$$

The fermionic local operators $\gamma^i \in H^3$ correspond to the first descendant of the operator $\mathcal{O}$ integrated over 1-cycles c_i on C:

$$\gamma_i = \int_{c_i} (\chi^i \chi^{\bar{k}} \chi^{\bar{l}} \; \mathrm{Tr}\Phi_{i\bar{k}} \alpha_{w\bar{l}} dw + \chi^i \chi^{\bar{k}} \chi^j \; \mathrm{Tr}\Phi_{i\bar{k}} \alpha_{\bar{w}j} d\bar{w}). \tag{2.12}$$

All the rest of the physical operators in the 4D topological YM theory become the descendants of these operators.[4]

Consider now the dimensional reduction of the $N=4$ SYM theory. It is convenient to consider the *partially*-twisted version of this theory. In the complex notation the bosonic content of the model is the following: the gauge field A, two complex scalar bosons ϕ ($\bar{\phi}$) and φ ($\bar{\varphi}$) and $(1,0)$ and $(0,1)$ forms on C denoted by ϕ_w and $\phi_{\bar{w}}$ respectively[5]. These non-scalar bosonic fields appear because twisting is performed with conserved current that includes a bosonic contribution. The fermionic fields are doublets with respect to the $SU(2)$ global group which is the unbroken subgroup of $SU(4)$ corresponding to $N=4$

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[4] Of course in the topological σ-model there are more physical operators. However the above listed operators generate all the rest.

[5] The twist that we use is the partial twisting corresponding to that used in ref.[6].

supersymmetry (the BRST charges are doublets with respect to this global group). There are the following fermionic fields: two scalars (on C) η^a_- and λ^a_-, $(1,0)$ and $(0,1)$ forms λ^a_{w-} and $\bar{\lambda}^a_{\bar{w}-}$, two vectors represented (after contracting with metric) by χ^a_{w+}, $\bar{\chi}^a_{\bar{w}+}$, and additional scalars on C denoted by χ^a_+ and $\bar{\chi}^a_+$. Here the indices $a = 1,2$ correspond to the doublet representation of the unbroken $SU(2)$ global group, the vector indices w and $\bar{w}$ correspond to the surface C, and the indices $\pm$ stand for (right-) left-handed spinors indices on Σ.

The dimensional reduction here is slightly different from that of above cases in the following respects. First, some of the bosonic fields which are scalar fields in the untwisted theory become 1-forms in the twisted model. Therefore their kinetic term is not suppressed as $\epsilon \to 0$ and may still correspond to propagating degrees of freedom in the dimensionally reduced theory. Second, there are unsuppressed terms in the Lagrangian which describe ϕ^4 interactions of the bosonic fields. In the limit $\epsilon \to 0$ the equations of motion reduce to

$$F_{w\bar{w}} = -i[\phi_{\bar{w}}, \phi_w], \quad D_w \phi_{\bar{w}} = 0, \quad D_{\bar{w}} \phi_w = 0. \tag{2.13}$$

This set of equations coincides with the Hitchin's equations for 'stable pairs' [7]. The moduli space $\mathcal{M}^H$ of solutions to Hitchin equations is the target space of supersymmetric 2d σ-model. Roughly speaking, one may think of Hitchin's space as partial compactification of the cotangent bundle to the moduli space of flat connections. As expected, $\mathcal{M}^H$ turns out to be a hyperKähler manifold of dimension $dim_{\mathbf{C}} \mathcal{M}^H = 6g - 6$ [7]. This implies that σ-model has $(4,4)$ superconformal symmetry. These facts are discussed in appendix B[6].

3. Applications

3.1. Aspects of Target Spaces $\mathcal{M}$ and $\mathcal{M}^H$

In this section we make some comments about $\mathcal{M}$ and $\mathcal{M}^H$. For concreteness let us concentrate on $SU(2)/Z_2 = SO(3)$. We denote by g the genus of Σ and by h the genus of C. The SO(3) bundles on $\Sigma \times C$ are characterised by the instanton number (the first Pontryagin

[6] There are interesting generalizations of Hitchin equation when we consider other theories. For example for $N=1$ theories with $SU(N)$ gauge group and with $2N$ flavors a similar equation appears where there are N holomorphic ϕ's in the above equation appearing in the fundamental representation of $SU(N)$. This is a generalization of Vortex equations studied by mathematicians [8]. These generalizations of Hitchin space are currently under investigation.

class) and the ‘t Hooft magnetic flux (the second Stiefel -Whitney class) $z \in H^2(\Sigma\times C, \mathbf{Z}_2)$. The magnetic flux has $4gh+2$ components corresponding to decomposition $H^2(\Sigma\times C, \mathbf{Z}_2) = H^2(\Sigma, \mathbf{Z}_2) + H^1(\Sigma, \mathbf{Z}_2) \otimes H^1(C, \mathbf{Z}_2) + H^2(C, \mathbf{Z}_2)$. To avoid complications as discussed before in this paper we mainly consider gauge field configurations with nonzero magnetic flux $z(C) = 1$ through the “small” surface C. Then $\mathcal{M}$ can be essentially identified with the space of representations of the group $\tilde{\pi}_1(C) = \langle a_i,\, b_i \,|\, \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1} = -1\rangle$ in $SU(2)$. (An unusual -1 in the right hand side appears exactly because of the nonzero flux through C.) This space is smooth and compact. Note that flipping the signs of a_i and b_i does not change the element they correspond to on $SO(3)$. Thus to be more precise the space that appears for the target space in the $SO(3)$ theory is an *orbifold* of this space. We consider on $\mathcal{M}$ a group $G = H^1(C, \mathbf{Z}_2)$ which acts by flipping the signs of a_i and b_i. This action is not free and each element $\alpha \in G$ fixes an $(h-1)$-dimensional complex torus $T_\alpha^{2(h-1)}$. The σ-model which corresponds to 4d $N = 2$ SYM has as a target space $\mathcal{M}/G$. The path-integral sectors of the orbifold σ-model on the worldsheet Σ_g are classified by the boundary conditions on A_i and B_i cycles of Σ_g, i.e. by the elements of $H^1(\Sigma, G) \approx H^1(\Sigma, \mathbf{Z}_2) \otimes H^1(C, \mathbf{Z}_2)$. As one can see in the 4d language by using the definition of ‘t Hooft magnetic flux the boundary conditions are equivalent to the choice of $4gh$ components of the magnetic flux z in $H^1(\Sigma, \mathbf{Z}_2) \otimes H^1(C, \mathbf{Z}_2)$. There remains a flux $z(\Sigma)$ through the worldsheet which is either zero or one. Apriori since we have not fixed it, we expect that the σ-model sums over both allowed values. The instanton number p of the SYM theory is $p = -z^2/4 \bmod 1$ differs by a factor of 2 from the σ-model instanton number $k = 2p = -z(\Sigma) + const$, where *const* depends on the orbifold subsector. Therefore turning on and off the $z(\Sigma)$ for each orbifold subsector shifts the parity of the instanton number. We can thus isolate the contributions corresponding to $z(\Sigma)$ on and off in the 2d σ model.

The same story repeats for the Hitchin space $\mathcal{M}^H$. The σ-model appearing in the physical theory is an orbifold of this space corresponding to the group G described above. In fact $\mathcal{M}^H$ may be viewed [7] as a certain partial compactification of $T^*\mathcal{M}$ and the relevant sigma model is the quotient of this space by G.

3.2. N=2 Application

We now discuss the implications of the above reduction for the $N=2$ YM theory. In the $N=2$ case in the fully twisted version we can use the computation of Donaldson invariants in the 4d theory to compute quantum cohomology ring on $\mathcal{M}$[7].

The classical cohomology of $\mathcal{M}$ is well studied both by mathematical [10][11][12] and physical [9] methods. The ring $H^*(\mathcal{M})$ is generated by the elements a, $\{\gamma_i\}_{i=1}^{2h}$, b of degrees 2, 3 and 4 respectively. In our language these generators appear, as discussed before, as $a=\int_C [\mathrm{Tr}\,\phi^2]^{(2)}$, $\gamma_i=\int_{c_i}[\mathrm{Tr}\,\phi^2]^{(1)}$ and $b=\mathrm{Tr}\,\phi^2$. Note that although *nonlocal* in 4d SYM, the operators a and γ_i become *local* after the reduction to 2d σ-model. The modular group of C acts on γ_i. The intersection numbers (correlation functions) are modular-invariant, hence they can be computed in terms of the modular-invariant subring $H^*(\mathcal{M})^{inv}$, generated by a, b and $c=\sum J_{ij}\gamma_i\gamma_j$ where $J_{ij}=c_i\cap c_j$ is the intersection form on C. There are three relations $R_1^h=0, R_2^h=0, R_3^h=0$ in $H^*(\mathcal{M})^{inv}$ in degrees $2h$, $2h+2$, $2h+4$ respectively. And the relations for $h+1$ can be expressed in terms of ones for h by the recursion relations of [13], [14]: $R_1^{h+1}=aR_1^h+h^2R_2^h$, $R_2^{h+1}=bR_1^h+\frac{2h}{h+1}R_3^h$, $R_3^{h+1}=cR_1^h$. (Formally $R_1^1=a, R_2^1=b, R_3^1=c$.)

The chiral ring of the topological σ-model on $\mathcal{M}$ is a quantum deformation of $H^*(\mathcal{M})$. It suffices to find the deformation of $H^*(\mathcal{M})^{inv}$ because it gives the full information about the modular invariant correlation functions of the σ-model. This deformation is generated by a, b, c and $q=\exp(2\pi i\tau)$, where q counts the instantons and has a formal degree 4 (because $2c_1(\mathcal{M})=4$). There are three relations $Q_1^h=0$, $Q_2^h=0$ and $Q_3^h=0$ which are reduced to $R_1^h=0$, $R_2^h=0$ and $R_3^h=0$ for $q=0$. We will be able to find the quantum relations using the results of 4d Donaldson theory. But first we need to describe more precisely the correspondence between 4d $SO(3)$ SYM and 2d σ-model.

The group G acts trivially on $H^*(\mathcal{M})$. Thus the chiral ring of the *untwisted* sector of orbifold $\mathcal{M}/G$ is the same as the chiral ring of the σ-model on $\mathcal{M}$. It allows us to

[7] If we consider the case $C=S^2$, the gauge fields on C would be frozen and thus we obtain an effective theory on Σ which is just the $N=2$ YM. In this case the Donaldson observable of 4d get mapped to Donaldson observables of the 2d YM which has been shown [9] to compute *classical* cohomology ring of $\mathcal{M}(\Sigma)$. If, on the other hand, we take S^2 to be large and Σ to be small, as we have argued, we obtain the *quantum* cohomology ring of $\mathcal{M}(\Sigma)$. This is not a contradiction, as it is known that Donaldson theory is anomalous when $b_2^+=1$ and the topological amplitudes depend on the choice of metric, as is the case here. Similarly, in the context of $N=4$ the topological reduction on S^2 gives $N=4$ YM on Σ which should thus enjoy S-duality.

draw conclusions about the quantum ring of that σ-model from the correlation function $\sum_z \langle e^{\alpha a+\beta b}\rangle_z$ computed in $SO(3)$ Donaldson theory on $\Sigma\times C$. Two magnetic fluxes corresponding to untwisted boundary conditions have zero components in $H^1(\Sigma,\mathbf{Z}_2)\otimes H^1(C,\mathbf{Z}_2)$ and differ by the value of $z(\Sigma)$. When $\Sigma=T^2$ is a complex 1-torus,

$$\sum_{z(\Sigma)}\langle e^{\alpha a+\beta b}\rangle_z^{T^2\times C}=\mathrm{Str}_{\mathcal{H}}e^{\alpha a+\beta b} \tag{3.1}$$

computes the (weighted with signs) sum over the spectrum of a and b. In principle, because of the signs, the contribution of some eigenvalues could be cancelled off completely. We make a minimal assumption that it does not happen and that we can read the whole spectrum off $\mathrm{Str}_{\mathcal{H}}e^{\alpha a+\beta b}$. The correlation function $\langle e^{\alpha a+\beta b}\rangle_z^{T^2\times C}$ can be obtained using the results of [15] or [16][8]:

$$\begin{aligned}\langle e^{\alpha a+\beta b}\rangle_z^{T^2\times C}=(-1)^{z(\Sigma)(h-1)}\big[&e^{-\lambda\beta}(e^{-\alpha x}-(-1)^{z(\Sigma)}e^{\alpha x})^{2h-2}+\\ &i^{-z^2}e^{\lambda\beta}(e^{-i\alpha x}-(-1)^{z(\Sigma)}e^{i\alpha x})^{2h-2}\big]/2\end{aligned} \tag{3.2}$$

where $\lambda=8q$ and $x=2i\sqrt{q}$ (these normalizations are consistent with the one used by Donaldson [17]). Summing (3.2) over two values of $z(\Sigma)$ one sees that b has two eigenvalues $\pm\lambda$ and a has $2g-1$ eigenvalues $\pm(0,\,2ix,\,4x,\,12ix,\,16x,\,\ldots)$. As for the third generator c, since it is bilinear in fermions it is nilpotent ($c^{g+1}=0$) so the only eigenvalue is 0. The spectrum together with the condition that the quantum ring relations are q-deformations of the classical ones determine the relations $Q_1^h,\ Q_2^h,\ Q_3^h$ completely. We put formally $Q_1^1=a, Q_2^1=b-8q, Q_3^1=c$. A straightforward analysis shows that the recursion relations are modified in a simple way:

$$Q_1^{h+1}=aQ_1^h+h^2Q_2^h,\quad Q_2^{h+1}=(b+(-1)^{h-1}8q)Q_1^h+\frac{2h}{h+1}Q_3^h,\quad Q_3^{h+1}=cQ_1^h \tag{3.3}$$

In particular, for genus 2 the relations are $Q_1^2=a^2+b-8q,\ \ Q_2^2=ba+8qa+c,\ \ Q_3^2=ca$. Up to a redefinition of generators $a\to h_2$, $b\to -4h_4+4q$, $c\to 4(h_6-qh_2)$ these relations

[8] In fixing the overall normalization we have to be a bit careful: Taking into account that we are discussing the $SO(3)$ as opposed to $SU(2)$ case modifies the overall coefficient of [15][16] by a factor of 2^{-b_1} where $b_1=2(g+h)$. In addition in the σ-model at genus g we have the orbifold symmetry factor $1/(\mathrm{dim}G)^g=2^{-2gh}$; So we have to multiply the overall normalization of [15][16] by $2^{-2g-2h+2gh}$, which up to a redefinition of the string coupling constant on Σ leaves us with a factor of 2^{-1}.

coincide with those found by Donaldson (for $g=2$) [17], where h_2, h_4, h_6 are generators of *integral* cohomology. Already this example shows that at the quantum level the σ-model operator $b = \mathrm{Tr}\phi^2$ is shifted by $\propto q{\cdot}1$ and the operator c is shifted by $\propto q{\cdot}a$ from the rational cohomology generators they used to be classically. As discussed before this possibility is allowed as in going from classical to quantum identification we had to consider composite operators and the definition of composite operator may receive instanton correction.

The simplicity of (3.3) should not be misleading. The very existence of quantum deformation consistent with the Donaldson spectrum is by no means obvious. Moreover, one can find the full quantum cohomology ring starting from the modular invariant subring defined by (3.3). Taking into account the fermionic contributions (odd cohomology) one can compute the partition function on a torus ($\mathrm{Str}_{\mathcal{H}} e^{\alpha a+\beta b}$). We explicitly checked the relation (3.1) for small genus $g = 2,3$. For example for $g = 3$ the one loop partition function

$$\begin{aligned}\mathrm{Str}_{\mathcal{H}} e^{\alpha a+\beta b} =& \Big(2e^{\lambda\beta}(e^{2\alpha ix}+e^{-2\alpha ix})+e^{-\lambda\beta}(18+e^{4\alpha x}+e^{-4\alpha x})\Big)- \\ & \Big(6e^{\lambda\beta}(e^{2i\alpha x}+e^{-2i\alpha x})+12e^{-\lambda\beta}\Big)\end{aligned} \tag{3.4}$$

These two brackets are bosonic and fermionic contributions respectively. 14 out of the 18 bosonic states corresponding to eigenvalues ($b=-\lambda, a=0$) correspond to the bilinears in fermions.

For the worldsheets Σ_g of genus $g>1$ the 4d SYM [16] gives the answer

$$\begin{aligned}\langle e^{\alpha a+\beta b}\rangle_z^{\Sigma_g\times C} =& (-1)^{z(\Sigma)(h-1)}\big[e^{-\lambda\beta}(e^{-\alpha x(h-1)}+(-1)^{(g-1)(h-1)}e^{\alpha x(h-1)}) \\ & +i^{-z^2}e^{\lambda\beta}(e^{i\alpha x(h-1)}+(-1)^{(g-1)(h-1)}e^{i\alpha x(h-1)})\big]/2\end{aligned} \tag{3.5}$$

One may find the derivation of this formula in Appendix C. Taking the sum over $z(\Sigma_g) = 0,1$ one sees that only two out of $2h-1$ eigenvalues of a contribute, those with the maximal absolute value $4(h-1)\sqrt{q}$. This is consistent with the fact that only for these values of a the above ring relations give non-degenerate eigenvalues of operators (a,b,c). The higher genus amplitude is obtained by inserting the handle operator H raised to the power of $g-1$, and the handle operator is essentially the mass operator which vanishes at degenerate points as there is no mass gap. If $g>1$ at the degenerate eigenvalues H^{g-1} vanishes leaving us with the contribution of the two non-degenerate eigenvalues.

Having described the untwisted sector of our orbifold σ-model, we can turn to the twisted sectors. The Hilbert space $\mathcal{H}_\alpha$ of the α-twisted sector, $\alpha \in G$, consists of G-invariant part of the cohomology of the complex $(h-1)$-dimensional torus $T_\alpha^{2(h-1)}$ fixed

by α. The fermionic numbers of all the vectors from $\mathcal{H}_\alpha$ should be shifted up by $2(h-1)$ to match those from the untwisted sector. The group G acts on $T_\alpha^{2(h-1)}$ by translations and reflections, so the G-invariant part of cohomology coincide with the even part. Finally, $\mathcal{H}_\alpha = H_{even}^{*+2(h-1)}(T_\alpha^{2(h-1)})$ and $\dim\mathcal{H}_\alpha = 2^{2h-3}$. Note that since $\mathcal{H}_\alpha$ is bosonic its dimension is computed by the Witten index $\mathrm{Str}_{\mathcal{H}_\alpha}1$. One can check that the 4d computation gives just the right answer 2^{2h-3} (the computation involves summation over 2^{2h} fluxes z necessary to project onto G-invariant states). In fact we can do better, namely we can examine the contribution of each twisted path-integral sector. The orbifold partition function on torus with the boundary conditions along A, B cycles twisted by $\mathcal{A},\mathcal{B}\in G$ computes, in absolute value, the number $F_{\mathcal{A},\mathcal{B}}$ of points fixed by both $\mathcal{A}$ and $\mathcal{B}$. The formula (3.2) leads to $F_{\mathcal{A},\mathcal{B}} = 0$ if $\mathcal{A}\cap\mathcal{B} = 0$ and $F_{\mathcal{A},\mathcal{B}} = 2^{2h-2}$ if $\mathcal{A}\cap\mathcal{B} = 1$ (we use here the cap product $\cap$ in $G = H^1(C,\mathbf{Z}_2)$ and the identification $H^2(C,\mathbf{Z}_2) = \mathbf{Z}_2$). This means that although the tori $T_\mathcal{A}^{2(h-1)}$ and $T_\mathcal{B}^{2(h-1)}$ have very small dimension — $2\dim T_\mathcal{A}^{2(h-1)} = 4(h-1) < 6(h-1) = \dim\mathcal{M}$ — they intersect one another! This surprising conclusion is true and can be explicitly checked using the definition of $\mathcal{M}$ as a space of representations of $\tilde{\pi}_1(C)$. In particular if $\mathcal{A}\cap\mathcal{B}=0$, $\mathcal{B}$ acts as translation on $T_\mathcal{A}^{2h-2}$ and if $\mathcal{A}\cap\mathcal{B}=1$ it acts as a total reflection giving 2^{2h-2} fixed points.

3.3. N=4 Application

Now we turn to the discussion of aspects of the reduced $N=4$ YM in two dimensions. As we discussed before the two dimensional theory we have obtained is a supersymmetric σ-model on the Hitchin space $\mathcal{M}^H$ which is a hyperKähler manifold. Since $\mathcal{M}^H$ is a smooth hyperKähler manifold the corresponding sigma model is a superconformal theory[9], which is in accord with the fact that one expects the four dimensional theory to be superconformal as well.

Since the coupling constant τ of the 4d YM theory gets identified with the unique complexified Kähler class for $\mathcal{M}^H$, the Montonen-Olive conjecture [18] for the 4d N=4 YM, gets translated to the modularity properties of the topological σ-model with respect to the Kähler moduli τ. In particular for $SU(N)$, the moduli space for τ should be a fundamental

[9] Just as in the $N=2$ case, for the $SO(3)$ theory the actual target for the σ-model is the quotient $\mathcal{M}^H/G$ which is also a nice superconformal theory if $\mathcal{M}^H$ is. Moreover one can easily extend the S-duality discussed in this section in the context of $\mathcal{M}^H$ to that for $\mathcal{M}^H/G$ which effectively has the matrix D discussed in this section replaced by D^{-1}. Aspects of the sharpened version of the S-duality conjecture can be verified in this context.

domain for the subgroup $\Gamma_0(N)$ (with lower off-diagonal entry being 0 mod N) of $SL(2, \mathbf{Z})$ (see [6] and note that $\Sigma \times C$ has even quadratic form on H^2). The S-duality conjecture in 4d thus gets translated to a T-duality for this 2d σ-model. However for σ-models we basically understand how T-duality may arise and thus we may be able to shed some light on the S-duality in 4d theories. We will show why the Hitchin's σ-model has T-duality. Before doing this let us see why this map of S-duality to T-duality is a resonable thing to expect.

In fact this is a natural generalization of the S-duality for the abelian $N = 4$ theory: If we consider $SU(2)$ gauge group and choose the internal space $C = T^2$, with a magnetic flux turned on, the σ-model becomes trivial (i.e. the Hitchin space is just a point). However if we don't turn on the flux, as discussed before we do not get a simple reduction to a $2d$ theory as different $4d$ field configurations lead to different regimes of the reduced theory which are connected to each other in a complicated way. In one field regime which corresponds to large expectation values for ϕ, i.e. the Higgs phase, the theory reduces to a $U(1)$ gauge theory plus a σ-model on the corresponding Hitchin space which in this case is just the cotangent of the moduli of flat connection (i.e. the cotangent of the torus which characterizes the holonomy of the unbroken $U(1)$ along the T^2 modulo the Weyl action). In other words, as noted in [19], the piece of the partition function compactified on T^2, which grows like the volume of ϕ, can be easily extracted from this complicated effective theory and is manifestly S-dual since for large ϕ the S-duality for the non-abelian theory gets mapped to S-duality for the abelian theory. In this context the field configurations which wrap around the σ-model torus get mapped to 4d field configurations where there is a magnetic flux for this unbroken $U(1)$ and the momentum modes are the dual configurations which are identified with the electrical flux of the unbroken $U(1)$. Thus the S-duality of the abelian theory gets mapped to T-duality[10]. Note, however, that it would be incorrect to ignore the other field configurations which make contributions to the path-integral which do not grow like the volume of ϕ. In fact it is relatively easy to see that ignoring those would lead to a Witten index for the σ-model which does not agree with that for the 4d theory (which for $SU(2)$ is 1 for the σ-model and 10 [6] for the 4d theory). Thus to make a really non-abelian test of S-duality we turn to the case where genus of C is greater than 1 and with 't Hooft magnetic flux turned on.

[10] The fact that in this context the S-duality is equivalent to the T-duality of toroidal compactification of the reduced theory has been independently noted in a recent paper [20].

There is a description of $\mathcal{M}^H$ which is most suitable for us [7]: For any gauge group G, $\mathcal{M}^H$ is a fiber space over the complex space $\mathbf{C}^d$ where $d = \mathrm{dimG}(h-1)$, whose fiber is a complex torus with complex dimension d. The complex structure of the torus varies holomorphically as we move in $\mathbf{C}^d$, but the Kähler structure of the torus is fixed and can be identified with the Kähler structure of $\mathcal{M}^H$. As we move the base point we reach points where the fiber is a singular torus but the total space is still smooth. The situation is a generalization of the cosmic string solution constructed in [21] where the base was $\mathbf{C}^1$ and the fiber a complex one dimensional torus. The basic strategy there was to use adiabatic approximation, by viewing the complex moduli as massless fields in $\mathbf{C}^1$ and to construct a hyperKähler metric by adiabatically varying the complex structure but with a fixed Kähler structure of the torus. Since the Kähler moduli is fixed for each fiber, this means that the modular properties of the Kähler moduli we will obtain, as long as we can trust the adiabatic approximation, will still be the same as that for each fiber (as the massless fields corresponding to varying it are turned off). The adiabatic approximation breaks down in the regions where the fiber becomes singular–however as was the case in [21] and as is the case for Hitchin space the total space is still a smooth hyperKähler space and we thus obtain an exact $(4,4)$ superconformal theory. Even though we may not have trusted adiabatic approximation for obtaining exact solutions, we do trust it as far as symmetries are concerned. Thus the Kähler moduli τ which can be identified with that of a non-singular fiber still enjoys the same modular properties as that of each fiber. Thus to find the modular properties of the Kähler parameter τ for $\mathcal{M}^H$ we have to study the modular properties of the Kähler modulus of the fiber torus.

Let us briefly explain why $\mathcal{M}^H$ has this toroidal fiber structure. For simplicity let us concentrate on $G = SU(2)$. Let $b_{ww} = \det\phi_w = -\frac{1}{2}\mathrm{Tr}\phi_w^2$. Then by Hitchin equations (2.13), $\overline{\partial} b_{ww} = 0$ whose solution can be identified with $\mathbf{C}^{3h-3}$, i.e. the complex $3h-3$ dimensional space. Generically a point of $\mathbf{C}^{3h-3}$ will correspond to a b_{ww} with isolated zeroes. Let us concentrate on such a solution. Away from the zeroes of b_{ww}, ϕ_w determines a $U(1)$ subspace of $SU(2)$, by the condition that $\Lambda = \phi_w/\sqrt{b_{ww}} = \pm 1$–more precisely we obtain a line bundle on the double cover $\hat{C}$ of C, which has genus $4h-3$, branched over the zeroes of b_{ww}. Away from the branch points the gauge field restricted to this $U(1)$ part is flat as follows from the fact that $\mathrm{Tr}F(\Lambda \pm 1) = 0$ because $\mathrm{Tr}F = 0$ and $\mathrm{Tr}F\phi_w = \mathrm{Tr}[\phi_w, \phi_{\overline{w}}]\phi_w = 0$. This line bundle will have delta function singularities at the branch points that can be gotten rid of by tensoring with a fixed line bundle with opposite singularity. The possible solutions to the Hitchin equation will thus give rise to flat bundles

on $\hat{C}$ which are parametrized by the Jacobian of $\hat{C}$, which can be viewed as the allowed holonomies of the $U(1)$ gauge group through the cycles of $\hat{C}$. However the allowed fluxes are parametrized by the Prym subspace of the Jacobian, which is the $3h-3$ dimensional complex torus which is odd under the Z_2 involution. This is because the involution on $\hat{C}$ exchanges the line bundle with its dual. We have thus given the description of $\mathcal{M}^H$ as a toroidal fiber space over C^{3h-3}. The generalization to $SU(N)$ is straightforward, with the base space being replaced by the space of allowed holomorphic differentials $Tr\phi_w^j$, where $j = 2, ..., N$, and by the fiber being the Prym variety of an N-fold cover of C [22]. Note that the S-duality getting mapped to T-duality of this fiber torus is a very natural generalization of what appears in the abelian case discussed above. Moreover it suggests an approach to showing S-duality for the non-abelian four dimensional theory by slicing the 4d path-integral in such a way that it becomes equivalent to a family of abelian S-dualities glued together in a nice way.

To get the precise form of the duality we thus have to study the moduli space of a complex d dimensional torus. The moduli space of a $2d$ real dimensional torus is known [23] to be

$$\frac{SO(2d,2d)}{SO(2d)\times SO(2d)\times SO(2d,2d;\mathbf{Z})}$$

If we fix an integral Kähler form $k \in H^2(T^{2d};\mathbf{Z})$ on the torus and ask about the moduli of complex structures on the torus with that fixed Kähler class the answer is described as follows[24] : Let $x^i, y^i, i = 1, ..., d$ denote the coordinates of torus with periodicity 1 in each direction which are chosen so that the Kähler form can be written as

$$k = \sum_{i=1}^{d} n_i dx^i \wedge dy^i \tag{3.6}$$

where n_i are positive integers. Let D denote the $d \times d$ diagonal matrix $D = (n_1, ..., n_d)$. Let z_i be the complex coordinates of the torus. Then we can choose them so that

$$dz^i = n_i dx^i + \sum_j \Omega_{ij} dy^j \tag{3.7}$$

where Ω is a complex, symmetric $d \times d$ matrix with a positive definite imaginary part (all follow from the fact that k defined above be a positive $(1,1)$ form) . We have $k = dz^i(\frac{1}{-2iIm\Omega})_{i\bar{j}}dz^{\bar{j}}$. We are interested in how the moduli space of complex structure and the particular complexified Kähler structure (rescaling the fixed Kähler class by t plus turning

on an anti-symmetric b field in the direction of the fixed Kähler form) imbed in the Narain moduli space. There is an action of symplectic group $Sp_J(2d)$ preserving the symplectic form

$$J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

on the moduli of complex structure, and the full moduli space of complex structures is given by the quotient $Sp_J(2d)/U(d) \times Sp_J(2d;\mathbf{Z})$, where $U(d)$ rotates the z_i among themselves. Note that $Sp_J(2d)$ is equivalent (and conjugate) to the canonical group as far as they are defined over the reals, but the group $Sp_J(2d;\mathbf{Z})$ very much depends on J (for example it would have been trivial if n_i were generic real numbers).

We will now show that the relevant moduli space for our problem is split to the complex and Kähler directions, where we just described the complex part. Since the Narain moduli space is described as a group quotient, it suffices to show that the generators of the complex deformations and the particular Kähler deformation commute. Let us first work over the real numbers, in which case we can rescale coordinates so that D is replaced by the identity matrix and J has the canonical form. It is not difficult to see that the generators of the deformations are then given by

$$\text{Complex}: \qquad (\sigma_x \otimes S; 1 \otimes A)/1 \otimes A$$

$$\text{Kahler}: \qquad (t = \sigma_x \otimes 1;\ b = i\sigma_y \otimes J;\ \sigma_z \otimes J)/\sigma_z \otimes J$$

where S and A denote symmetric and anti-symmetric generators of $Sp(2d)$, and the Pauli matrices act on the (L,R) decomposition of the Narain momenta. Note that the generators of Kähler deformations commute with those of complex deformations and form the $Sp(2)$ (or $SL(2)$) group. In fact this is the maximal subgroups of $SO(2d,2d)$ which commutes with $Sp(2d) \subset SO(2d,2d)$. In order to find how the modular group acts on the $Sp(2)$, given its imbedding in the Narain moduli, all we have to do is to find integral points of the group generated by $\sigma_x \otimes 1, i\sigma_y \otimes J, \sigma_z \otimes J$; We also have to recall that we have rescaled coordinates so that J is in the canonical form. If we undo this rescaling and we decompose $J = \oplus J_i$ where J_i corresponds to i-th direction corresponding to n_i, we can view our $Sp(2)$ as sitting diagonally in $\otimes Sp_i(2)$ where the common moduli τ is identified as $n_i\tau_i$ in each subfactor. With no loss of generality let us assume n_i's have no common divisor (otherwise rescale the Kähler form so this is true). Let n denote the least common multiple of n_i. Then it is clear that the common intersection of all the $SL_i(2,Z)$ is generated by T and

ST^nS where $S:\tau\to -1/\tau$ and $T:\tau\to\tau+1$. This generates the group $\Gamma_0(n)$. We thus have the moduli space

$$\frac{Sp(2d)}{U(d)\times Sp_J(2d,\mathbf{Z})}\times\frac{Sp(2)}{U(1)\times\Gamma_0(n)}$$

Thus the Kähler moduli of the Hitchin space has $\Gamma_0(n)$ as a modular group. For $SU(N)$, all the n_i are either N or 1, corresponding to whether they are related to combinations of cycles of $\hat{C}$ which are projected to trivial or non-trivial cycles of C. So in this case $n=N$ and we recover the prediction of the S-duality that τ should belong to the fundamental domain of $\Gamma_0(N)$. In fact there is more information in the modular transformation. In particular prediction of S-duality for $\tau\to -1/\tau$ is in accord with the relation between the Hitchin spaces for $SU(N)$ vs. $SU(N)/Z_N$.

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Appendix A. Aspects of $N=2$ Reduction

Let us consider some properties of the basis $\{\alpha\}$ and the connection E. Define a covariant derivative

$$\nabla_i=\partial/\partial X^i-iE_i\ ,\quad \nabla_{\bar{i}}=\partial/\partial X^{\bar{i}}-iE_{\bar{i}}\ ,$$

which acts on the space of Lie algebra valued functions on $\mathcal{M}$. By using eqs. (2.4), it is easy to check that $\partial_iA_w=D_wE_i$ and $\partial_{\bar{k}}A_{\bar{w}}=D_{\bar{w}}E_{\bar{k}}$, i.e. $[\nabla_i,D_w]=0=[\nabla_{\bar{i}},D_{\bar{w}}]$. The only non-zero component of the curvature is $(1,1)$

$$\Phi_{i\bar{j}}=i[\nabla_i,\nabla_{\bar{j}}]\ ,$$

so that ∇ is holomorphic. It is also easy to check that $\nabla_i \alpha_{w\bar{k}} = D_w \Phi_{i\bar{k}}$, $\nabla_{\bar{k}} \alpha_{\bar{w}i} = -D_{\bar{w}} \Phi_{i\bar{k}}$ and $D_i \alpha_{\bar{w}j} = D_j \alpha_{\bar{w}i}$, $D_{\bar{i}} \alpha_{w\bar{j}} = D_{\bar{j}} \alpha_{w\bar{i}}$. The Christoffel connections Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}}$ can be constructed in terms of the basis $\alpha_{\bar{w}i}$, $\alpha_{w\bar{k}}$ as follows

$$\Gamma_{ij,\bar{k}} = \int_C \mathrm{Tr}\ \alpha_{w\bar{k}} \nabla_j \alpha_{\bar{w}i} = \partial_j G_{i\bar{k}}, \quad \Gamma_{\bar{i}\bar{j},k} = \int_C \mathrm{Tr}\ \alpha_{\bar{w}k} \nabla_{\bar{i}} \alpha_{w\bar{j}} = \partial_{\bar{i}} G_{k\bar{j}}. \tag{A.1}$$

Notice that the other components vanish since the metric is Kähler. The basis vectors $\alpha_{\bar{w}i}$ ($\alpha_{w\bar{k}}$) are covariantly constant with respect to the covariant derivatives $\nabla_i \delta^k_j - \Gamma^k_{ij}$ ($\nabla_{\bar{i}} \delta^{\bar{k}}_{\bar{j}} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}}$) which act on Lie algebra valued 1-forms on $\mathcal{M}$, i.e. $\nabla_i \alpha_{\bar{w}j} = \Gamma^k_{ij} \alpha_{\bar{w}k}$ and $\nabla_{\bar{i}} \alpha_{w\bar{j}} = \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \alpha_{w\bar{k}}$. The latter equations follow from the fact that $D_w(\nabla_i \alpha_{\bar{w}j}) = 0 = D_{\bar{w}}(\nabla_{\bar{k}} \alpha_{w\bar{j}})$.

The Riemann tensor can also be written down in terms of the basis vectors as follows

$$R_{i\bar{k}j\bar{l}} = \int_C (D_i \alpha_{w\bar{k}}\ D_{\bar{l}} \alpha_{\bar{w}j} + D_j \alpha_{w\bar{k}}\ D_{\bar{l}} \alpha_{\bar{w}i}). \tag{A.2}$$

For convenience we also discuss briefly the equations of motion for A_Σ and ϕ. These equations of motion read

$$D_w F_{\bar{z}\bar{w}} = -i\{\lambda_{\bar{z}\bar{w}}, \chi_w\}, \quad D_{\bar{w}} F_{zw} = i\{\lambda_{zw}, \chi_{\bar{w}}\}, \tag{A.3}$$

and

$$D_w D_{\bar{w}} \phi = i\{\chi_w, \chi_{\bar{w}}\}, \quad D_{\bar{w}} D_w \bar{\phi} = i g^{z\bar{z}} \{\lambda_{zw}, \lambda_{\bar{z}\bar{w}}\}. \tag{A.4}$$

We assume that the connection A_C is irreducible and therefore the solutions of equations (A.3) and (A.4) are unique. At first glance the solutions to these equations are expected to be non-local. However, by using an identity $[\alpha_{\bar{w}i}, \alpha_{w\bar{k}}] = i D_{\bar{w}} D_w \Phi_{i\bar{k}}$ one can reduce the solutions to a local form. In particular for ϕ and $\bar{\phi}$ we get eq.(2.8).

Appendix B. Aspects of $N = 4$ Reduction

The cotangent vectors δA_C, $\delta\phi_w$ and $\delta\phi_{\bar{w}}$ to the moduli space $\mathcal{M}^H$ obey the equations which are variation of eq.(2.13). To study the moduli space of the Hitchin's equations it is convenient to choose a special gauge

$$D^w \delta A_w + D^{\bar{w}} \delta A_{\bar{w}} + i[\delta\phi_{\bar{w}}, \phi_w] + i[\phi_{\bar{w}}, \delta\phi_w] = 0. \tag{B.1}$$

Thus the cotangent vectors obey the following equations

$$D^w \delta A_w = -i[\delta\phi_{\bar{w}}, \phi_w], \quad D^{\bar{w}} \delta\phi_{\bar{w}} = i[\delta A_w, \phi_{\bar{w}}]. \tag{B.2}$$

To introduce explicitly the collective (real) coordinates $\{X^A\}$ it is convenient to choose a basis $\{(\alpha_{A\bar{w}}, \beta_{Aw})\}$ and $\{(\alpha_{Aw}, \beta_{A\bar{w}})\}$ on the space of pairs $(\delta A_w, \delta\phi_{\bar{w}})$ and $(\delta A_{\bar{w}}, \delta\phi_w)$. Then the variation $(\delta A_w, \delta\phi_{\bar{w}})$, $(\delta A_{\bar{w}}, \delta\phi_w)$ can be decomposed along the basis modulo a gauge transformation as follows

$$\partial_A A_w = \alpha_{Aw} + D_w E_A, \quad \partial_A A_{\bar{w}} = \alpha_{A\bar{w}} + D_{\bar{w}} E_A, \tag{B.3}$$

$$\partial_A \phi_w = \beta_{Aw} - i[\phi_w, E_A], \quad \partial_A \phi_{\bar{w}} = \beta_{A\bar{w}} - i[\phi_{\bar{w}}, E_A],$$

where $\partial_A = \partial/\partial X^A$, and E_A can be identified with a connection on $\mathcal{M}^H$. The moduli space $\mathcal{M}^H$ is hyperKähler and has a natural hyperKähler metric

$$G_{AB} = \int_C \mathrm{Tr}(\alpha_{Aw}\alpha_{B\bar{w}} + \beta_{Aw}\beta_{B\bar{w}} + (A \leftrightarrow B)), \tag{B.4}$$

induced by a bilinear form on the space of pairs $(\delta A_\alpha, \delta\Phi_\alpha)$ ($\alpha = 1, 2$ are Lorentz indices on C^h)

$$g((\delta A, \delta\Phi), (\delta A, \delta\Phi)) = \int_C \mathrm{Tr}(\delta A \wedge *\delta A + \delta\Phi \wedge *\delta\Phi), \tag{B.5}$$

where and $\phi_w = \Phi_1 + i\Phi_2$, $\phi_{\bar{w}} = \Phi_1 - i\Phi_2$. Similarly we introduce a Kähler form

$$\Omega^I_{AB} = \int_C \mathrm{Tr}(\alpha_{Aw}\alpha_{B\bar{w}} + \beta_{Aw}\beta_{B\bar{w}} - (A \leftrightarrow B)), \tag{B.6}$$

on $\mathcal{M}^H$ induced by a symplectic form

$$\omega^I((\delta_1 A, \delta_1\Phi), (\delta_2 A, \delta_2\Phi)) = \int_C \mathrm{Tr}(\delta_1 A \wedge \delta_2 A + \delta_1\Phi \wedge \delta_2\Phi). \tag{B.7}$$

It is easy to check that this form is closed. We naturally define the complex structure $I^A_B = G^{AC}\Omega^I_{BC}$ on $\mathcal{M}^H$.

With a complex structure I one can choose the complex coordinates X^μ and $X^{\bar{\mu}}$ on $\mathcal{M}^H$. It is easy to check that the only non-vanishing components of the basis vectors are $(\alpha_{\mu\bar{w}}, \beta_{\mu w})$ and $(\alpha_{\bar{\mu}w}, \beta_{\bar{\mu}\bar{w}})$.

Since the moduli space is hyperKähler there are two more complex structures J and K which satisfy the algebraic identities for the quaternions. One of these complex structures,

J, can be defined as $J^A_B = G^{AC}\Omega^J_{BC}$, where Ω^J_{BC} is a symplectic form on $\mathcal{M}^H$, which is induced by a symplectic form

$$\omega^J((\delta_1 A, \delta_1\Phi), (\delta_2 A, \delta_2\Phi)) = \int_C \mathrm{Tr}(\delta_1 A \wedge \delta_2\Phi - \delta_1\Phi \wedge \delta_2 A). \tag{B.8}$$

The symplectic form Ω^J has only non-vanishing components

$$\Omega^J_{\mu\nu} = \int_C \mathrm{Tr}(\alpha_{\mu\bar{w}}\beta_{\nu w} - \alpha_{\nu\bar{w}}\beta_{\mu w}) \quad \text{and} \quad \Omega^J_{\bar{\mu}\bar{\nu}} = \int_C \mathrm{Tr}(\alpha_{\bar{\mu}w}\beta_{\bar{\nu}\bar{w}} - \alpha_{\bar{\nu}w}\beta_{\bar{\mu}\bar{w}}), \tag{B.9}$$

which are anti-holomorphic and holomorphic on the moduli space $\mathcal{M}^H$, respectively, and obey the following equation

$$\Omega^J_{\mu\nu} G^{\nu\bar{\nu}} \Omega^J_{\bar{\mu}\bar{\nu}} = G_{\mu\bar{\mu}}. \tag{B.10}$$

Similar to the analysis of the moduli space of flat connections we see that $[\nabla_\mu, D_w] = 0$, $[\nabla_{\bar{\mu}}, D_{\bar{w}}] = 0$, $[\nabla_\mu, \phi_{\bar{w}}] = 0$ and $[\nabla_{\bar{\mu}}, \phi_w] = 0$, where $\nabla_\mu = \partial_\mu - iE_\mu$ and $\nabla_{\bar{\mu}} = \partial_{\bar{\mu}} - iE_{\bar{\mu}}$. One can also easily check that $[\nabla_\mu, \nabla_\nu] = 0$ and $[\nabla_{\bar{\mu}}, \nabla_{\bar{\nu}}] = 0$, and hence $\nabla_\mu \alpha_{\bar{w}\nu}$ ($\nabla_{\bar{\mu}}\alpha_{w\bar{\nu}}$) and $\nabla_\mu \phi_{w\nu}$ ($\nabla_{\bar{\mu}}\phi_{\bar{w}\bar{\nu}}$) are symmetric with respect to indices $\mu,\ \nu$ ($\bar{\mu},\ \bar{\nu}$). This follows from the fact that the forms on $\mathcal{M}^H$ $\Phi_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$ and $\Phi_{\bar{\mu}\bar{\nu}} = i[\nabla_{\bar{\mu}}, \nabla_{\bar{\nu}}]$ are annihilated by the operator $D_{\bar{w}}D_w - [\phi_w, [\phi_{\bar{w}}, \cdot]]$ (we assume that a non-trivial magnetic flux through C^h is turned on).

It is also worth noticing that the complex structure J exchanges the α and β components of the basis as follows

$$J^\mu_{\bar{\mu}}\alpha_{\bar{w}\mu} = \beta_{\bar{w}\bar{\mu}}, \quad J^\mu_{\bar{\mu}}\beta_{w\mu} = -\alpha_{w\bar{\mu}}, \quad J^{\bar{\mu}}_\mu \alpha_{w\bar{\mu}} = \beta_{w\mu}, \quad J^{\bar{\mu}}_\mu \beta_{w\bar{\mu}} = -\alpha_{w\mu}. \tag{B.11}$$

As a generalization of the $N = 2$ case we also have

$$\nabla_\nu \alpha_{\bar{w}\mu} = \Gamma^\lambda_{\mu\nu}\alpha_{\bar{w}\lambda} - iJ^{\bar{\mu}}_\mu[\Phi_{\nu\bar{\mu}}, \phi_{\bar{w}}], \quad \nabla_\nu \beta_{w\mu} = \Gamma^\lambda_{\mu\nu}\beta_{w\lambda} - J^{\bar{\mu}}_\mu D_w \Phi_{\nu\bar{\mu}}, \tag{B.12}$$

and similar relations for $\alpha_{w\bar{\mu}}$ and $\beta_{\bar{w}\bar{\mu}}$. Here $\Phi_{\mu\bar{\mu}} = i[\nabla_\mu, \nabla_{\bar{\mu}}]$, and $\Gamma^\lambda_{\mu\nu}$ stand for the Christoffel connection which is defined as follows

$$\Gamma_{\mu\nu,\bar{\lambda}} = \int_C (\alpha_{w\bar{\lambda}}\nabla_\mu \alpha_{\bar{w}\nu} + \beta_{\bar{w}\bar{\lambda}}\nabla_\mu \beta_{w\nu}) = \partial_\mu G_{\nu\bar{\lambda}}, \tag{B.13}$$

$$\Gamma_{\bar{\mu}\bar{\nu},\lambda} = \int_C (\alpha_{\bar{w}\lambda}\nabla_{\bar{\mu}}\alpha_{w\bar{\nu}} + \beta_{w\lambda}\nabla_{\bar{\mu}}\beta_{\bar{w}\bar{\nu}}) = \partial_\mu G_{\lambda\bar{\nu}}.$$

One may wonder if eqs.(B.12) are consistent with the symmetry of the Christoffel connection. It is easy to see that the consistency condition reads $J^{\bar\mu}_{\nu}\Phi_{\mu\bar\mu}J^{\mu}_{\bar\nu}=-\Phi_{\nu\bar\nu}$.

Let us continue our discussion of the reduction for $N=4$ in the case of partially twisted model (in C-direction). By substituting the solutions to the Hitchin's equations for A_C, ϕ_w and $\phi_{\bar w}$ into the action one can see that in the limit $\epsilon\to 0$ the rest of the bosonic fields enter quadratically into the action. They are scalar on C^h and therefore do not correspond to any propagating degrees of freedom in the effective 2D σ-model provided the fields A_C, ϕ_w and $\phi_{\bar w}$ correspond to irreducible configuration (for example, one can consider the $SO(3)$ gauge group with a non-trivial flux through C^h). By the equations of motion these bosonic fields are reduced to certain combinations of the fermionic fields similar to the $N=2$ case. The solutions of the equations of motion for the scalar fields (on C^h) can be expressed in local form similar to those of the $N=2$ case due to the following equation

$$D_{\bar w}D_w\Phi_{\mu\bar\mu}-[\phi_w,[\phi_{\bar w},\Phi_{\mu\bar\mu}]]=i[\alpha_{w\bar\mu},\alpha_{\bar w\mu}]+i[\beta_{\bar w\bar\mu},\beta_{w\mu}]. \tag{B.14}$$

Integrating out the fermionic fields which are scalars on C^h one gets constraints on the fermionic fields which have a vector index on C^h. By these constraints the latter fields are naturally split into pairs $(\chi^a_{w+},\bar\lambda^a_{\bar w-})$, $(\chi^a_{\bar w+},\lambda^a_{w-})$ which are enforced to obey the eqs.(B.2) for cotangent vectors on the moduli space $\mathcal{M}^H$, and hence can be decomposed along the above chosen basis. Thus these fields give rise to 2d fermionic fields ψ^μ_+, $\psi^{\bar\mu}_+$, ψ^μ_- and $\psi^{\bar\mu}_-$. Finally for the dimensionally reduced theory we get the standard action for a supersymmetric 2d sigma-model

$$\begin{aligned}S^H=\frac{1}{e^2}\int_\Sigma \mathrm{d}^2z\; G_{\mu\bar\mu}(\frac{1}{2}\partial_zX^\mu\;\bar\partial_{\bar z}X^{\bar\mu}+\frac{1}{2}\bar\partial_{\bar z}X^\mu\;\partial_zX^{\bar\mu}+\\ \psi^\mu_+\overline{D}_{\bar z}\psi^{\bar\mu}_+ +\psi^\mu_-D_z\psi^{\bar\mu}_-)-R_{\mu\bar\mu\nu\bar\nu}\psi^{\bar\mu}_+\psi^\mu_-\chi^{\bar\nu}_-\psi^\mu_+.\end{aligned} \tag{B.15}$$

Here the covariant derivatives D_z and $\overline{D}_{\bar z}$ are constructed by pulling back the Christoffel connection on the tangent bundle $T\mathcal{M}^{\mathcal{H}}$ to $\mathcal{M}^{\mathcal{H}}$.

The contributions from the path integral over quadratic fluctuations orthogonal to the zero modes cancel due to supersymmetry. Notice also that the fermion number current is non-anomalous similar to the one which appears in unreduced 4d $N=4$ supersymmetric Yang-Mills theory.

We can also consider the totally twisted 4d N=4 Yang-Mills theory. By the dimensional reduction (with the twist used in ref.[6]) we get the Lagrangian of a twisted version of the above supersymmetric σ-model. The twisting current has a bosonic piece and hence

some of the bosonic fields become 1-forms on the world sheet Σ. In 4-dimensional theory this current generates $U(1)$ global phase rotations of ϕ and $\bar{\phi}$. In terms of the *partially* twisted theory the bosonic contributions to the current is $j_n = \mathrm{Tr}(\bar{\phi}^w\ \partial_n\phi_w - \partial_n\bar{\phi}^w\ \phi_w + ...)$, where n is a worldsheet index on Σ. Under the dimensional reduction this current becomes

$$j_n = \int_C \mathrm{Tr}(\bar{\phi}_{\bar{w}}(X)\partial_n\phi_{\bar{w}}(X) - \partial_n\bar{\phi}_{\bar{w}}(X)\phi_w(X)) + (\text{fermionic terms}), \tag{B.16}$$

where $X(z,\bar{z})$ determines the map $\Sigma \to \mathcal{M}^{\mathcal{H}}$. The fields $\bar{\phi}_{\bar{w}}$ and ϕ_w obey the Hitchin's equations and hence are functions on $\mathcal{M}^H$. In fact, in the σ-model j_n is a Noether current corresponding to the action of $U(1)$ on $\mathcal{M}^H$ given by $(A,\phi) \to (A, e^{i\theta}\phi)$. This group action is Poisson with respect to the symplectic form ω^I (B.7) and the hamiltonian $\mu = \int_C \mathrm{Tr}\,\bar{\phi}_{\bar{w}}\phi_w$. By the equations of motion of the σ-model j_n is conserved.

Actually the above splitting of the coordinates on $\mathcal{M}^H$ naturally appears under the dimensional reduction of the totally twisted 4d N=4 SYM theory. In this case we start with Hitchin's fields ϕ_{zw} and $\phi_{\bar{z}\bar{w}}$ which are 2-forms on M. Under the dimensional reduction we have to assign the worldsheet indices $z, \bar{z}$ to some of the components of the basis vectors, and hence to some of collective coordinates.

To understand better this twisting procedure recall that $\mathcal{M}^H$ is fibered over $\mathbf{C}^{3g-3}$ with $\det\phi_w$ projecting onto the base [7]. On the base $\mathbf{C}^{3g-3}$ let us introduce the standard affine coordinates $Y^1, \cdots, Y^{3g-3}$, which constitute half of $6g-6$ coordinates on the full space $\mathcal{M}^H$. In such a description the $U(1)$ generated by (B.16) acts by the phase rotation $Y^I \to e^{i\theta}Y^I$. Let us denote by Z^i the coordinates along fiber, which are inert under the $U(1)$ rotations. In this coordinate system the action looks as follows

$$\begin{aligned} S^H = \frac{1}{2e^2}\int_\Sigma \mathrm{d}^2 z\ \Big(& G_{i\bar{k}}(\partial_z Z^i\ \bar{\partial}_{\bar{z}} Z^{\bar{k}} + \bar{\partial}_{\bar{z}} Z^i\ \partial_z Z^{\bar{k}}) + \\ & G_{I\bar{I}} g^{z\bar{z}}(D_z Y^I_z\ \overline{D}_{\bar{z}} Y^{\bar{I}}_{\bar{z}} + \overline{D}_{\bar{z}} Y^I_z\ D_z Y^{\bar{I}}_{\bar{z}}) + \\ & C_{i\bar{I},J} Y^J_z g^{z\bar{z}}(\bar{\partial}_{\bar{z}} Z^i\ D_z Y^{\bar{I}}_{\bar{z}} + \partial_z Z^i\ \overline{D}_{\bar{z}} Y^{\bar{I}}_{\bar{z}}) + \\ & C_{I\bar{k},\bar{I}} Y^{\bar{I}}_{\bar{z}} g^{z\bar{z}}(\overline{D}_{\bar{z}} Y^I_z\ \partial_z Z^{\bar{k}} + D_z Y^I_z\ \bar{\partial}_{\bar{z}} Z^{\bar{k}}) + \\ & \qquad \text{fermion terms}\Big)\,, \end{aligned} \tag{B.17}$$

where $C_{i\bar{I},J}Y^J = G_{i\bar{I}}$, $C_{I\bar{k},\bar{I}}Y^{\bar{I}} = G_{I\bar{k}}$. All the terms in the action have equal number of Y^I_z and $Y^{\bar{I}}_{\bar{z}}$ which are contracted by appropriate number of $g^{z\bar{z}}$. This is the action for a new topological σ-model, which is nothing but $N=4$ superconformal theory twisted

with a current discussed above. This theory computes the Euler class of moduli of holomorphic maps to Hitchin space, as is clear from similar constructions in the literature [25][26][27][28][29][30][31][32][6]. In this context using the results of [6], we have a prediction for the Euler characteristic of moduli of holomorphic maps from Σ to $\mathcal{M}^H(C)$. A similar construction has been considered in the context of $N=2$ string with cotangent of a Riemann surface as the target [33].

Appendix C.

Here we would like to sketch the derivation of formulas (3.2), (3.5) computing the correlation function $\langle e^{\alpha a+\beta b}\rangle_z^{M^4}$ of $N=2$ topological supersymmetric Yang Mills theory in a sector with fixed 't Hooft flux z. To do it, one can use either the "cosmic strings" [15] or the Monopole Equation [16]. The four-manifold M^4 for us is either $T^2\times C^h$ or $\Sigma\times C^h$, where both genera g and h satisfy $g,h>1$. For the case $M^4=T^2\times C^h$ the canonical divisor consists of $2g-2$ *nonintersecting* components ("cosmic strings"). When $M^4=\Sigma\times C^h$ the cosmic string is a Riemann surface of genus $8(g-1)(h-1)+1$. Indeed, the canonical class of $\Sigma\times C^h$ is very ample so one can choose its divisor to be a smooth connected curve. It follows from Bertini's theorem [34] Ch. 2, Prop. 8.18 and the standard technical lemma [34] Ch. 3, Prop.7.9. Then the formula (2.79) from [15] gives (3.2) and (3.5) (we need to fix the right normalization though, see the footnote after (3.2)).

The other way to obtain (3.2) and (3.5) is through the Monopole Equation [16]. The arguments in Section 4 there tell us that the basic class $x\in H^2(\Sigma\times C^h)$ should satisfy the following conditions: 1) It comes from $H^2(\Sigma,\mathbf{Z})+H^2(C^h,\mathbf{Z})$: $x=x(\Sigma)\omega_\Sigma+x(C^h)\omega_C$ and is divisible by 2: $x(\Sigma)\equiv 0(\mathrm{mod}2)$, $x(C^h)\equiv 0(\mathrm{mod}2)$; 2) $x^2=4(g-1)(h-1)$; 3) $1-g\leq x(\Sigma)/2\leq g-1$ and $1-h\leq x(C^h)/2\leq h-1$. These constraints fix $x=\pm(2(g-1)\omega_\Sigma+2(h-1)\omega_C)$ to be (plus or minus) the canonical class K. The correlation function is given by [16]

$$\langle e^{\alpha a+\beta b}\rangle_z^{\Sigma\times C^h}=\frac{1}{2}\sum_{x=\pm K}(-1)^{z\cdot x/2}n_x(e^{\alpha tx(C)+\beta\lambda}+i^{z^2}e^{-i\alpha tx(C)-\beta\lambda}) \tag{C.1}$$

where t is the normalization constant and n_x is "multiplicity". The absolute value of n_x computes the number of decompositions $\alpha\beta=\eta$ where η is a fixed holomorphic 2-differential — a section of K, α is a section of the line bundle $\mathcal{L}\otimes K^{\frac{1}{2}}$ and β is a section of the line bundle $\mathcal{L}^{-1}\otimes K^{\frac{1}{2}}$. The bundle $\mathcal{L}$ should satisfy $2c_1(\mathcal{L})=x$. Since $x=\pm K$, either

$\mathcal{L} = K^{\frac{1}{2}}$ and then $\alpha = \eta$, $\beta = 1$ or $\mathcal{L} = K^{-\frac{1}{2}}$ and $\alpha = 1$, $\beta = \eta$. In both cases, $|n_x| = 1$. The sign of n_x can be obtained by the cohomological computation suggested in [16] which gives $n_K = 1$ and $n_{-K} = (-1)^{(g-1)(h-1)}$ which finally leads to (3.5) in the main text.

Using the Monopole Equation one can also verify the results of [15] for $M^4 = T^2 \times C^h$. The same constraints 1)–3) single out the basic classes $\{x = 2j\omega_C \,|1 - h \leq j \leq h - 1\}$. The correlation function $\langle e^{\alpha a + \beta b} \rangle_z^{T^2 \times C^h}$ is computed by the equation like (C.1), only now one has to sum over all x from the list above. Also in this case the line bundle $\mathcal{L}$ comes only from the curve C. So $|n_x|$ computes the number of ways to split 1-differential η *on* C as a product of two sections. For $\deg\mathcal{L} = j$, the section α has $j + h - 1$ zeroes and β has $h - 1 - j$ zeroes. Together they form $2h - 2$ zeroes of η. Since the divisor of zeroes determine the line bundle on C completely, *each* splitting of zeroes of η into two groups of $j + h - 1$ and $h - 1 - j$ elements gives a solution so $|n_x| = C^{2h-2}_{j+h-1}$. Also, the computation gives $\mathrm{sgn}\,(n_x) = (-1)^{j+h-1}$ and we arrive at (3.2).

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